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Technical Memorandum

STATISTICAL THEOREMS INVOLVING THE COMPLEX MULTIVARIATE GAUSSIAN AND WISHART DENSITIES

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The complex multivariate Gaussian and Wishart densities are often encountered in the analysis of statistical signal processing algorithms in frequency domain array processing. The derivation of the probability density function of forms involving complex Gaussian random vectors and complex Wishart matrices may be simplified by the use of standard theorems found in multivariate statistics for real Gaussian random vectors and real Wishart matrices. This memorandum contains the extensions to the complex case of a selection of these theorems.

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Abstract

The *complex* multivariate Gaussian and Wishart densities are often encountered in the analysis of statistical signal processing algorithms in frequency domain array processing. The derivation of the probability density function of forms involving complex Gaussian random vectors and complex Wishart matrices may be simplified by the use of standard theorems found in multivariate statistics for *real* Gaussian random vectors and *real* Wishart matrices. This memorandum contains the extensions to the complex case of a selection of these theorems.

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STATISTICAL THEOREMS INVOLVING THE COMPLEX MULTIVARIATE GAUSSIAN AND WISHART DENSITIES

INTRODUCTION

The theorems found in this memorandum are extensions to the complex case of theorems commonly found in the analysis of multivariate real Gaussian random variables. The original theorems may be found in either Searle [1] or Muirhead [2]. The extensions are, for the most part, straightforward and result from following the proofs for the real cases. When possible, the proofs have been simplified or deferred. Background and distributional information on multivariate complex Gaussian random variables and complex Wishart matrices may be found in Goodman [3] and to a lesser extent in Anderson [4].

NOTATION

In this memorandum, boldface lower case letters represent vectors, boldface upper case letters represent matrices, and the superscript ^{H} represents the conjugate transpose operation. If the n dimensional random vector $\mathbf{x}$ has a complex Gaussian distribution, the notation

$$\mathbf{x} \sim \mathcal{CN}_n(\mu, \mathbf{\Sigma}) \tag{1}$$

is used where μ is the mean vector and $\mathbf{\Sigma}$ is the covariance matrix. The probability density function of $\mathbf{x}$ is

$$f(\mathbf{x}) = \frac{1}{\pi^n |\mathbf{\Sigma}|} e^{-(\mathbf{x}-\mu)^H \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}. \tag{2}$$

If the n-by-n random matrix $\mathbf{A}$ has a complex Wishart distribution, the notation

$$\mathbf{A} \sim \mathcal{CW}_n(k, \mathbf{\Sigma}) \tag{3}$$

is used where k is the number of degrees of freedom and $\mathbf{\Sigma}$ is the scale matrix. The probability density function of the complex Wishart distribution is described in Theorem 3. If the scalar random variable Z has a non-central chi-squared distribution, the notation

$$Z \sim \mathcal{X}_p^2(\delta) \tag{4}$$

is used, where P is the number of degrees of freedom and δ is the non-centrality parameter. The probability density function of such a non-central chi-squared random variable is, as found in Johnson and Kotz [5],

$$f(z) = \sum_{k=0}^{\infty} \frac{e^{-\frac{\delta}{2}} \left(\frac{\delta}{2}\right)^k}{k!} \frac{z^{\frac{P+2k}{2}-1} e^{-\frac{z}{2}}}{2^{\frac{P+2k}{2}} \Gamma\left(\frac{P+2k}{2}\right)}. \tag{5}$$

It should be noted that Searle [1] defines the non-central chi-squared probablity density function as having a non-centrality parameter equal to $\frac{\delta}{2}$ in equation (5).

THEOREM 1

The following theorem is an extension of Theorem 2, Chapter 2, page 57 of Searle [1] describing the probability density function of certain quadratic forms involving complex Gaussian random vectors.

Theorem 1 *If* $\mathbf{x} \sim \mathcal{CN}_n(\mu, \mathbf{\Sigma})$ *and* $\mathbf{A} = \mathbf{A}^H$, *then* $2\mathbf{x}^H\mathbf{A}\mathbf{x} \sim \mathcal{X}_r^2(\delta)$ *if and only if* $\mathbf{A\Sigma}$ *is idempotent. The non-centrality parameter,* $\delta = 2\mu^H\mathbf{A}\mu$, *and the degrees of freedom parameter,* $r = 2tr(\mathbf{A\Sigma})$, *where* $tr(\mathbf{A\Sigma})$ *is the trace of* $\mathbf{A\Sigma}$.

Proof: For the quadratic form

$$Q = 2\mathbf{x}^H\mathbf{A}\mathbf{x}, \tag{6}$$

consider the moment generating function

$$\begin{aligned} M_Q(t) &= \mathrm{E}\left[e^{tQ}\right] \\ &= \mathrm{E}\left[e^{2t\mathbf{x}^H\mathbf{A}\mathbf{x}}\right] \\ &= \int_{\mathcal{C}^n} \frac{1}{\pi^n |\mathbf{\Sigma}|} \exp\left[2t\mathbf{x}^H\mathbf{A}\mathbf{x} - (\mathbf{x}-\mu)^H \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)\right] d\mathbf{x} \\ &= \int_{\mathcal{C}^n} \frac{1}{\pi^n |\mathbf{\Sigma}|} e^V d\mathbf{x}, \end{aligned} \tag{7}$$

where $\mathcal{C}^n$ represents n dimensional complex Euclidean space. The exponent, V, may be massaged into a single quadratic form in $\mathbf{x}$,

$$V = \mathbf{x}^H(2t\mathbf{A})\mathbf{x} - \mathbf{x}^H\mathbf{\Sigma}^{-1}\mathbf{x} + \mu^H\mathbf{\Sigma}^{-1}\mathbf{x} + \mathbf{x}^H\mathbf{\Sigma}^{-1}\mu - \mu^H\mathbf{\Sigma}^{-1}\mu$$

$$= \; -\left(\mathbf{x}-\tilde{\mu}\right)^H \tilde{\mathbf{\Sigma}}^{-1}\left(\mathbf{x}-\tilde{\mu}\right) + \tilde{\mu}^H \tilde{\mathbf{\Sigma}}^{-1}\tilde{\mu} - \mu^H \mathbf{\Sigma}^{-1}\mu, \tag{8}$$

where

$$\tilde{\mathbf{\Sigma}}^{-1} = \mathbf{\Sigma}^{-1} - 2t\mathbf{A} \tag{9}$$

and

$$\tilde{\mu} = \tilde{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\mu. \tag{10}$$

The moment generating function thus becomes,

$$\begin{aligned} M_Q(t) &= \frac{\exp\left[\tilde{\mu}^H \tilde{\mathbf{\Sigma}}^{-1}\tilde{\mu} - \mu^H \mathbf{\Sigma}^{-1}\mu\right]}{\left|\tilde{\mathbf{\Sigma}}\right|^{-1}\left|\mathbf{\Sigma}\right|} \int_{C^n} \frac{\exp\left[-\left(\mathbf{x}-\tilde{\mu}\right)^H \tilde{\mathbf{\Sigma}}^{-1}\left(\mathbf{x}-\tilde{\mu}\right)\right]}{\pi^n \left|\tilde{\mathbf{\Sigma}}\right|} d\mathbf{x} \\ &= \frac{\exp\left[\mu^H \left(\mathbf{\Sigma}^{-1}\tilde{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\right)\mu\right]}{\left|\tilde{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma}\right|} \\ &= \frac{\exp\left[-\mu^H \left(\mathbf{I}_n - \mathbf{\Sigma}^{-1}\left(\mathbf{\Sigma}^{-1} - 2t\mathbf{A}\right)^{-1}\right)\mathbf{\Sigma}^{-1}\mu\right]}{\left|\mathbf{I}_n - 2t\mathbf{A}\mathbf{\Sigma}\right|} \\ &= \frac{\exp\left[-\mu^H \left(\mathbf{I}_n - \left(\mathbf{I}_n - 2t\mathbf{A}\mathbf{\Sigma}\right)^{-1}\right)\mathbf{\Sigma}^{-1}\mu\right]}{\left|\mathbf{I}_n - 2t\mathbf{A}\mathbf{\Sigma}\right|}, \end{aligned} \tag{11}$$

where $\mathbf{I}_n$ is the n dimensional identity matrix.

Suppose that $\mathbf{A\Sigma}$ is idempotent with rank or trace,

$$\operatorname{tr}\left(\mathbf{A\Sigma}\right) = m. \tag{12}$$

Note that the eigenvalues of an idempotent matrix are either zero or one, and, by using a singular value decomposition, the matrix $\mathbf{A\Sigma}$ may be expressed as

$$\mathbf{A\Sigma} = \mathbf{U}\mathbf{V}^H, \tag{13}$$

where the n-by-m matrices $\mathbf{U}$ and $\mathbf{V}$ are orthogonal,

$$\mathbf{V}^H\mathbf{U} = \mathbf{I}_m. \tag{14}$$

Part of the matrix in the quadratic form in the exponent of equation (11) may now be simplified,

$$\mathbf{I}_n - \left(\mathbf{I}_n - 2t\mathbf{A\Sigma}\right)^{-1} \; = \; \mathbf{I}_n - \left(\mathbf{I}_n - \mathbf{U}\left(2t\right)\mathbf{V}^H\right)^{-1}$$

$$
\begin{aligned}
&= \mathbf{I}_n - \left[\mathbf{I}_n - \mathbf{U}\left(\mathbf{V}^H\mathbf{U} - \frac{1}{2t}\mathbf{I}_m\right)^{-1}\mathbf{V}^H\right] \\
&= \mathbf{U}\left(1 - \frac{1}{2t}\right)^{-1}\mathbf{V}^H \\
&= \left(1 - \frac{1}{1-2t}\right)\mathbf{A\Sigma}. \qquad (15)
\end{aligned}
$$

The determinant in the denominator of equation (11) is clearly

$$
|\mathbf{I}_n - 2t\mathbf{A\Sigma}| = (1-2t)^m \qquad (16)
$$

because $\mathbf{A\Sigma}$ has m non-zero eigenvalues, all equal to one. Substituting (15) and (16) into (11) results in

$$
\begin{aligned}
M_Q(t) &= \frac{\exp\left[-\left(1 - \frac{1}{1-2t}\right)\mu^H\mathbf{A}\mu\right]}{(1-2t)^m} \\
&= \mathrm{e}^{-\mu^H\mathbf{A}\mu}\frac{\exp\left(\frac{1}{1-2t}\mu^H\mathbf{A}\mu\right)}{(1-2t)^{\frac{2m}{2}}}. \qquad (17)
\end{aligned}
$$

The moment generating function of a non-central Chi-squared random variable with r degrees freedom and non-centrality parameter δ, as found in Muirhead [2], is

$$
M_{\chi^2_{r,\delta}}(t) = \frac{\mathrm{e}^{-\frac{\delta}{2}}\mathrm{e}^{\frac{\delta}{2(1-2t)}}}{(1-2t)^{\frac{r}{2}}}. \qquad (18)
$$

Equating equations (17) and (18), it is seen that the quadratic form, Q, has a non-central Chi-squared distribution with $r = 2m = 2\mathrm{tr}\,(\mathbf{A\Sigma})$ degrees of freedom and non-centrality parameter $\delta = 2\mu^H\mathbf{A}\mu$.

If it is assumed that the quadratic form has the described non-central Chi-squared distribution, it can be seen that the matrix $\mathbf{A\Sigma}$ must be idempotent with rank equal to $\mathrm{tr}\,(\mathbf{A\Sigma}) = \frac{r}{2}$ by equating the denominators of equations (11) and (18),

$$
\begin{aligned}
(1-2t)^{\frac{r}{2}} &= |\mathbf{I}_n - 2t\mathbf{A\Sigma}| \\
&= \prod_{i=1}^{n}(1 - 2t\lambda_i), \qquad (19)
\end{aligned}
$$

where λ_i are the eigenvalues of $\mathbf{A\Sigma}$. Clearly the left and right sides of equation (19) will be equal only when $\frac{r}{2}$ of the eigenvalues are equal to one with the remaining equal to zero, which results in an idempotent $\mathbf{A\Sigma}$ matrix.

THEOREM 2

The following theorem is the extension of Theorem 4, Chapter 2, page 59 of Searle [1] showing the independence of quadratic forms involving complex Gaussian random vectors.

Theorem 2 *If* $\mathbf{x} \sim \mathcal{CN}_n(\mu, \mathbf{\Sigma})$, $\mathbf{A} = \mathbf{A}^H$ *and* $\mathbf{B} = \mathbf{B}^H$, *then* $\mathbf{x}^H\mathbf{A}\mathbf{x}$ *and* $\mathbf{x}^H\mathbf{B}\mathbf{x}$ *are independent if and only if* $\mathbf{A\Sigma B} = \mathbf{B\Sigma A} = \mathbf{0}$.

Proof: The proof is identical to that of Searle [1] and, as the result is also identical, will not be repeated.

THEOREM 3

The following theorem is an extension of Theorem 3.2.10 on page 93 of Muirhead [2] describing the probability density functions of the partitions of a complex Wishart matrix and of a particular function of the partitions that enjoys certain independence properties.

Theorem 3 *If the n-by-n matrix* $\mathbf{A} \sim \mathcal{CW}_n(m, \mathbf{\Sigma})$ *with* $\mathbf{A}$ *and* $\mathbf{\Sigma}$ *partitioned into two-by-two blocks,*

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \\ \mathbf{\Sigma} &= \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix}, \end{aligned} \tag{20}$$

where $\mathbf{A}_{11}$ *and* $\mathbf{\Sigma}_{11}$ *are k-by-k, and if*

$$\mathbf{A}_{11\cdot 2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \tag{21}$$

and

$$\mathbf{\Sigma}_{11\cdot 2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}, \tag{22}$$

then

1. $\mathbf{A}_{11\cdot 2} \sim \mathcal{CW}_k(m - n + k, \mathbf{\Sigma}_{11\cdot 2})$ *and is independent of* $\mathbf{A}_{12}$ *and* $\mathbf{A}_{22}$.

2. The conditional distribution of $\mathbf{A}_{12}$ *given* $\mathbf{A}_{22}$ *is*

$$\mathcal{CN}_{k\times(n-k)}\left(\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{A}_{22}, \mathbf{\Sigma}_{11\cdot 2}\otimes\mathbf{A}_{22}\right). \tag{23}$$

3. $\mathbf{A}_{22} \sim \mathcal{CW}_{n-k}\left(m, \mathbf{\Sigma}_{22}\right)$.

Proof: Consider the probability density function of the complex Wishart distributed matrix, $\mathbf{A}$, as found in Goodman [3],

$$f(\mathbf{A}) = \frac{|\mathbf{A}|^{m-n}}{\Gamma_{m,n}|\mathbf{\Sigma}|^{m}}\mathrm{etr}\left(-\mathbf{\Sigma}^{-1}\mathbf{A}\right), \tag{24}$$

where 'etr' signifies the matrix exponential trace operation and

$$\Gamma_{m,n} = \pi^{\frac{n(n-1)}{2}}\Gamma(m)\cdots\Gamma(m-n+1) \tag{25}$$

where $\Gamma(x)$ is the standard Gamma function. Applying the k, $n-k$ factorization to the determinants of equation (24) yields

$$\begin{aligned} |\mathbf{A}| &= |\mathbf{A}_{22}|\left|\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right| \\ &= |\mathbf{A}_{22}|\left|\mathbf{A}_{11\cdot 2}\right| \end{aligned} \tag{26}$$

and, similarly,

$$|\mathbf{\Sigma}| = |\mathbf{\Sigma}_{22}|\left|\mathbf{\Sigma}_{11\cdot 2}\right|. \tag{27}$$

The determinant of a partitioned matrix may be found in the Matrix Theory Appendix of Muirhead [2]. The trace of the matrix product $\mathbf{\Sigma}^{-1}\mathbf{A}$ in equation (24) must now be massaged into a form containing the matrix partitions $\mathbf{A}_{12}$ and $\mathbf{A}_{22}$ and the matrix $\mathbf{A}_{11\cdot 2}$. This requires substantial algebraic manipulation, beginning with the matrix factorization of the inverse of the matrix $\mathbf{\Sigma}$,

$$\mathbf{\Sigma}^{-1} = \mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}. \tag{28}$$

The following relationships between the partitioned matrix and the partitioned inverse may be easily verified:

$$\mathbf{V}_{11} = \mathbf{\Sigma}_{11\cdot 2}^{-1} \tag{29}$$

$$\Sigma_{22}^{-1} = \mathbf{V}_{22} - \mathbf{V}_{21}\mathbf{V}_{11}^{-1}\mathbf{V}_{12} \tag{30}$$
$$\mathbf{V}_{11}^{-1}\mathbf{V}_{12} = -\Sigma_{12}\Sigma_{22}^{-1}. \tag{31}$$

Applying these to the aforementioned trace results in

$$\begin{aligned}
\mathrm{tr}\left(\Sigma^{-1}\mathbf{A}\right) &= \mathrm{tr}\left(\begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\right) \\
&= \mathrm{tr}\left(\begin{bmatrix} \mathbf{V}_{11}\mathbf{A}_{11} + \mathbf{V}_{12}\mathbf{A}_{21} & \mathbf{V}_{11}\mathbf{A}_{12} + \mathbf{V}_{12}\mathbf{A}_{22} \\ \mathbf{V}_{21}\mathbf{A}_{11} + \mathbf{V}_{22}\mathbf{A}_{21} & \mathbf{V}_{21}\mathbf{A}_{12} + \mathbf{V}_{22}\mathbf{A}_{22} \end{bmatrix}\right) \\
&= \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{11}\right) + \mathrm{tr}\left(\mathbf{V}_{12}\mathbf{A}_{21}\right) + \mathrm{tr}\left(\mathbf{V}_{21}\mathbf{A}_{12}\right) + \mathrm{tr}\left(\mathbf{V}_{22}\mathbf{A}_{22}\right) \\
&= \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{11}\right) + 2\mathrm{tr}\left(\mathbf{V}_{12}\mathbf{A}_{21}\right) + \mathrm{tr}\left(\mathbf{V}_{22}\mathbf{A}_{22}\right) \\
&= \mathrm{tr}\left(\mathbf{V}_{11}\left[\mathbf{A}_{11\cdot 2} + \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right]\right) + 2\mathrm{tr}\left(\mathbf{V}_{12}\mathbf{A}_{21}\right) + \mathrm{tr}\left(\mathbf{V}_{22}\mathbf{A}_{22}\right) \\
&= \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{11\cdot 2}\right) + \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right) + 2\mathrm{tr}\left(\mathbf{V}_{12}\mathbf{A}_{21}\right) + \mathrm{tr}\left(\mathbf{V}_{22}\mathbf{A}_{22}\right) \\
&= \left\{\begin{array}{c} \mathrm{tr}\left(\Sigma_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right) + \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right) \\ -2\mathrm{tr}\left(\mathbf{V}_{11}\Sigma_{12}\Sigma_{22}^{-1}\mathbf{A}_{21}\right) + \mathrm{tr}\left(\left[\Sigma_{22}^{-1} + \mathbf{V}_{21}\mathbf{V}_{11}^{-1}\mathbf{V}_{12}\right]\mathbf{A}_{22}\right) \end{array}\right\} \\
&= \left\{\begin{array}{c} \mathrm{tr}\left(\Sigma_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right) + \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right) - 2\mathrm{tr}\left(\mathbf{V}_{11}\Sigma_{12}\Sigma_{22}^{-1}\mathbf{A}_{21}\right) \\ +\mathrm{tr}\left(\Sigma_{22}^{-1}\mathbf{A}_{22}\right) + \mathrm{tr}\left(\mathbf{A}_{22}\Sigma_{22}^{-1}\Sigma_{21}\mathbf{V}_{11}\Sigma_{12}\Sigma_{22}^{-1}\right) \end{array}\right\} \\
&= \left\{\begin{array}{c} \mathrm{tr}\left(\Sigma_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right) + \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right) \\ -\mathrm{tr}\left(\mathbf{V}_{11}\Sigma_{12}\Sigma_{22}^{-1}\mathbf{A}_{21}\right) - \mathrm{tr}\left(\mathbf{V}_{11}\mathbf{A}_{12}\Sigma_{22}^{-1}\Sigma_{21}\right) \\ +\mathrm{tr}\left(\Sigma_{22}^{-1}\mathbf{A}_{22}\right) + \mathrm{tr}\left(\mathbf{V}_{11}\Sigma_{12}\Sigma_{22}^{-1}\mathbf{A}_{22}\Sigma_{22}^{-1}\Sigma_{21}\right) \end{array}\right\} \\
&= \left\{\begin{array}{c} \mathrm{tr}\left(\Sigma_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right) + \mathrm{tr}\left(\Sigma_{22}^{-1}\mathbf{A}_{22}\right) \\ +\mathrm{tr}\left(\mathbf{V}_{11}\begin{bmatrix} \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} - \Sigma_{12}\Sigma_{22}^{-1}\mathbf{A}_{22}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \\ -\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{22}\Sigma_{22}^{-1}\Sigma_{21} \\ +\Sigma_{12}\Sigma_{22}^{-1}\mathbf{A}_{22}\mathbf{A}_{22}^{-1}\mathbf{A}_{22}\Sigma_{22}^{-1}\Sigma_{21} \end{bmatrix}\right) \end{array}\right\} \\
&= \left\{\begin{array}{c} \mathrm{tr}\left(\Sigma_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right) + \mathrm{tr}\left(\Sigma_{22}^{-1}\mathbf{A}_{22}\right) \\ +\mathrm{tr}\left(\Sigma_{11\cdot 2}^{-1}\left[\Sigma_{12}\Sigma_{22}^{-1}\mathbf{A}_{22} - \mathbf{A}_{12}\right]\mathbf{A}_{22}^{-1}\left[\mathbf{A}_{22}\Sigma_{22}^{-1}\Sigma_{21} - \mathbf{A}_{21}\right]\right) \end{array}\right\}
\end{aligned}$$

$$= \left\{ \begin{array}{c} \operatorname{tr}\left(\boldsymbol{\Sigma}_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right) + \operatorname{tr}\left(\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right) \\ +\operatorname{tr}\left(\boldsymbol{\Sigma}_{11\cdot 2}^{-1}\left[\mathbf{A}_{12} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]\mathbf{A}_{22}^{-1}\left[\mathbf{A}_{12} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]^{H}\right) \end{array} \right\} \tag{32}$$

Substituting equations (26), (27), and (32) into the probability density function of equation (24) yields

$$\begin{aligned} f(\mathbf{A}) &= \left\{ \begin{array}{c} \left[\frac{|\mathbf{A}_{11\cdot 2}|^{m-n}}{\Gamma_{m,n}|\boldsymbol{\Sigma}_{11\cdot 2}|^{m}} \operatorname{etr}\left(-\boldsymbol{\Sigma}_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right)\right]\left[\frac{|\mathbf{A}_{22}|^{m-n}}{|\boldsymbol{\Sigma}_{22}|^{m}} \operatorname{etr}\left(-\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right)\right] \\ \cdot\operatorname{etr}\left(-\boldsymbol{\Sigma}_{11\cdot 2}^{-1}\left[\mathbf{A}_{12} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]\mathbf{A}_{22}^{-1}\left[\mathbf{A}_{12} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]^{H}\right) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \left[\frac{|\mathbf{A}_{11\cdot 2}|^{m-(n-k)-k}}{\Gamma_{m-(n-k),k}|\boldsymbol{\Sigma}_{11\cdot 2}|^{m-(n-k)}} \operatorname{etr}\left(-\boldsymbol{\Sigma}_{11\cdot 2}^{-1}\mathbf{A}_{11\cdot 2}\right)\right]\left[\frac{|\mathbf{A}_{22}|^{m-(n-k)}}{\Gamma_{m,n-k}|\boldsymbol{\Sigma}_{22}|^{m}} \operatorname{etr}\left(-\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right)\right] \\ \cdot\left[\begin{array}{c} \frac{\Gamma_{m-(n-k),k}\Gamma_{m,n-k}}{\Gamma_{m,n}} \frac{|\mathbf{A}_{22}|^{-k}}{|\boldsymbol{\Sigma}_{11\cdot 2}|^{n-k}} \\ \operatorname{etr}\left(-\boldsymbol{\Sigma}_{11\cdot 2}^{-1}\left[\mathbf{A}_{12} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]\mathbf{A}_{22}^{-1}\left[\mathbf{A}_{12} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]^{H}\right) \end{array} \right] \end{array} \right\} \\ &= f\left(\mathbf{A}_{11\cdot 2}\right) f\left(\mathbf{A}_{22}\right) f\left(\mathbf{A}_{12}|\mathbf{A}_{22}\right), \end{aligned} \tag{33}$$

where it is recognized that $\mathbf{A}_{11\cdot 2}$ and $\mathbf{A}_{22}$ have complex Wishart distributions,

$$\mathbf{A}_{11\cdot 2} \sim \mathcal{CW}_{k}\left(m - (n-k), \boldsymbol{\Sigma}_{11\cdot 2}\right) \tag{34}$$

and

$$\mathbf{A}_{22} \sim \mathcal{CW}_{n-k}\left(m, \boldsymbol{\Sigma}_{22}\right), \tag{35}$$

thus proving the first and third parts of the theorem. The independence of $\mathbf{A}_{11\cdot 2}$ from $\mathbf{A}_{12}$ and $\mathbf{A}_{22}$ is seen from the factorization of the probability density function of equation (33). The probability density function of $\mathbf{A}_{12}$ conditioned on $\mathbf{A}_{22}$ may be simplified by considering

$$\begin{aligned} \frac{\Gamma_{m-(n-k),k}\Gamma_{m,n-k}}{\Gamma_{m,n}} &= \left\{ \begin{array}{c} \frac{\Gamma(m-n+k)\cdots\Gamma(m-n+1)\cdot\Gamma(m)\cdots\Gamma(m-n+k+1)}{\Gamma(m)\cdots\Gamma(m-n+1)} \\ \cdot\pi^{\frac{1}{2}[k(k-1)+(n-k)(n-k-1)-n(n-1)]} \end{array} \right\} \\ &= \pi^{\frac{1}{2}\left[k^{2}-k+n^{2}-2nk+k^{2}-n+k-n^{2}+n\right]} \\ &= \pi^{-k(n-k)}. \end{aligned} \tag{36}$$

Substituting equation (36) into the conditional probability density function of $\mathbf{A}_{12}$ yields

$$f\left(\mathbf{A}_{12}|\mathbf{A}_{22}\right) = \frac{\operatorname{etr}\left(-\mathbf{\Sigma}_{11\cdot 2}^{-1}\left[\mathbf{A}_{12}-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]\mathbf{A}_{22}^{-1}\left[\mathbf{A}_{12}-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{A}_{22}\right]^{H}\right)}{\pi^{k(n-k)}\left|\mathbf{A}_{22}\right|^{k}\left|\mathbf{\Sigma}_{11\cdot 2}\right|^{n-k}} \tag{37}$$

which is recognized as the probability density function of a complex Gaussian random matrix,

$$\mathbf{A}_{12}|\mathbf{A}_{22} \sim \mathcal{CN}_{k\times(n-k)}\left(\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{A}_{22}, \mathbf{\Sigma}_{11\cdot 2}\otimes\mathbf{A}_{22}\right), \tag{38}$$

proving the second part of the theorem.

THEOREM 4

The following theorem is an extension of Theorem 3.2.11 on page 95 of Muirhead [2] describing the probability density function of the inverse of a matrix quadratic form involving the inverse of a complex Wishart distributed matrix.

Theorem 4 *If* $\mathbf{A} \sim \mathcal{CW}_n\left(m, \mathbf{\Sigma}\right)$, $\mathbf{\Sigma}$ *is full rank, and* $\mathbf{P}$ *is a* k*-by-*n *matrix with rank* k*, then*

$$\left(\mathbf{P}\mathbf{A}^{-1}\mathbf{P}^{H}\right)^{-1} \sim \mathcal{CW}_k\left(m-n+k, \left(\mathbf{P}\mathbf{\Sigma}^{-1}\mathbf{P}^{H}\right)^{-1}\right). \tag{39}$$

Proof: If $\mathbf{\Sigma}$ is full rank, it may be factored into

$$\mathbf{\Sigma} = \mathbf{\Gamma}\mathbf{\Gamma}^{H}, \tag{40}$$

where $\mathbf{\Gamma}$ is also of full rank. Set

$$\mathbf{B} = \mathbf{\Gamma}^{-1}\mathbf{A}\left(\mathbf{\Gamma}^{H}\right)^{-1}. \tag{41}$$

Since $\mathbf{\Gamma}$ is constant and of full rank, $\mathbf{B}$ is distributed as

$$\mathbf{B} \sim \mathcal{CW}_n\left(m, \mathbf{I}_n\right). \tag{42}$$

Define the k-by-n, rank k matrix

$$\mathbf{R} = \mathbf{P}\left(\mathbf{\Gamma}^H\right)^{-1}. \tag{43}$$

Substituting equations (43) and (41) into the matrix product described in the theorem results in

$$\begin{aligned}
\left(\mathbf{P}\mathbf{A}^{-1}\mathbf{P}^H\right)^{-1} &= \left(\mathbf{R}\mathbf{\Gamma}^H\mathbf{A}^{-1}\mathbf{\Gamma}\mathbf{R}^H\right)^{-1} \\
&= \left(\mathbf{R}\left[\mathbf{\Gamma}^{-1}\mathbf{A}\left(\mathbf{\Gamma}^H\right)^{-1}\right]^{-1}\mathbf{R}^H\right)^{-1} \\
&= \left(\mathbf{R}\mathbf{B}^{-1}\mathbf{R}^H\right)^{-1}.
\end{aligned} \tag{44}$$

Similarly,

$$\begin{aligned}
\left(\mathbf{P}\mathbf{\Sigma}^{-1}\mathbf{P}^H\right)^{-1} &= \left(\mathbf{R}\mathbf{\Gamma}^H\mathbf{\Sigma}^{-1}\mathbf{\Gamma}\mathbf{R}^H\right)^{-1} \\
&= \left[\mathbf{R}\left(\mathbf{\Gamma}^{-1}\mathbf{\Sigma}\left(\mathbf{\Gamma}^H\right)^{-1}\right)^{-1}\mathbf{R}^H\right]^{-1} \\
&= \left(\mathbf{R}\mathbf{R}^H\right)^{-1}.
\end{aligned} \tag{45}$$

Using equations (44) and (45), it is seen that the theorem, as stated in equation (39), simplifies to showing that

$$\left(\mathbf{R}\mathbf{B}^{-1}\mathbf{R}^H\right)^{-1} \sim \mathcal{CW}_n\left(m-n+k, \left(\mathbf{R}\mathbf{R}^H\right)^{-1}\right). \tag{46}$$

Using a singular value decomposition, the matrix $\mathbf{R}$ may be factored into

$$\mathbf{R} = \mathbf{U}\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix}\mathbf{V}^H, \tag{47}$$

where $\mathbf{U}$ is k-by-k and non-singular and $\mathbf{V}$ is n-by-n and orthogonal,

$$\mathbf{V}^{-1} = \mathbf{V}^H. \tag{48}$$

Substituting this factorization into equations (44) and (45) results in

$$\left(\mathbf{R}\mathbf{B}^{-1}\mathbf{R}^H\right)^{-1} = \left(\mathbf{U}\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix}\mathbf{V}^H\mathbf{B}^{-1}\mathbf{V}\begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}\mathbf{U}^H\right)^{-1}$$

$$
\begin{aligned}
&= \left(\mathbf{U}^H\right)^{-1} \left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix} \left(\mathbf{V}^H \mathbf{B} \mathbf{V}\right)^{-1} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} \right)^{-1} \mathbf{U}^{-1} \\
&= \left(\mathbf{U}^H\right)^{-1} \left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix} \mathbf{C}^{-1} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} \right)^{-1} \mathbf{U}^{-1},
\end{aligned} \tag{49}
$$

where

$$
\mathbf{C} = \mathbf{V}^H \mathbf{B} \mathbf{V}, \tag{50}
$$

and

$$
\begin{aligned}
\left(\mathbf{R}\mathbf{R}^H\right)^{-1} &= \left(\mathbf{U} \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix} \mathbf{V}^H \mathbf{V} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} \mathbf{U}^H \right)^{-1} \\
&= \left(\mathbf{U} \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} \mathbf{U}^H \right)^{-1} \\
&= \left(\mathbf{U}\mathbf{U}^H\right)^{-1}.
\end{aligned} \tag{51}
$$

Let

$$
\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \tag{52}
$$

and

$$
\mathbf{C}^{-1} = \mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} \tag{53}
$$

be k-by-$n-k$ partitions of $\mathbf{C}$ and $\mathbf{D} = \mathbf{C}^{-1}$. Then, equation (49) becomes

$$
\begin{aligned}
\left(\mathbf{R}\mathbf{B}^{-1}\mathbf{R}^H\right)^{-1} &= \left(\mathbf{U}^H\right)^{-1} \mathbf{D}_{11}^{-1} \mathbf{U}^{-1} \\
&= \left(\mathbf{U}^H\right)^{-1} \mathbf{C}_{11\cdot 2} \mathbf{U}^{-1},
\end{aligned} \tag{54}
$$

where $\mathbf{C}_{11\cdot 2}$ is as defined in Theorem 3 and, as seen in Muirhead [2], is equal to $\mathbf{D}_{11}^{-1}$. Since $\mathbf{V}$ is orthogonal, $\mathbf{C}$ is distributed as

$$
\mathbf{C} \sim \mathcal{CW}_n \left(m, \mathbf{I}_n\right). \tag{55}
$$

Applying part (i) of Theorem 3, it is seen that

$$
\mathbf{C}_{11\cdot 2} \sim \mathcal{CW}_k \left(m - n + k, \mathbf{I}_k\right), \tag{56}
$$

which, when applied to equation (54), results in

$$\begin{aligned}\left(\mathbf{R}\mathbf{B}^{-1}\mathbf{R}^{H}\right)^{-1} &\sim \mathcal{CW}_{k}\left(m-n+k,\left(\mathbf{U}\mathbf{U}^{H}\right)^{-1}\right)\\ &\sim \mathcal{CW}_{k}\left(m-n+k,\left(\mathbf{R}\mathbf{R}^{H}\right)^{-1}\right),\end{aligned} \tag{57}$$

which completes the proof.

THEOREM 5

The following theorem is an extension of Theorem 3.2.12 on page 96 of Muirhead [2] describing the probability density function of the ratio of quadratic forms involving the inverse of the scale matrix and the inverse of a random sample of a complex Wishart distribution. It is interesting to note that the resulting distribution does not depend on the vector in the quadratic forms if it is independent of the complex Wishart distributed matrix.

Theorem 5 *If* $\mathbf{A} \sim \mathcal{CW}_n(m,\mathbf{\Sigma})$ *where* $m > n-1$ *and if* $\mathbf{y}$ *is any* n*-by-*1 *random vector independent of* $\mathbf{A}$ *such that* $Pr(\mathbf{y}=\mathbf{0})=0$*, then*

$$2\frac{\mathbf{y}^{H}\mathbf{\Sigma}^{-1}\mathbf{y}}{\mathbf{y}^{H}\mathbf{A}^{-1}\mathbf{y}} \sim \mathcal{X}^{2}_{2(m-n+1)}, \tag{58}$$

and is independent of $\mathbf{y}$.

Proof: In Theorem 4, let $\mathbf{P}=\mathbf{y}^{H}$. Then,

$$W=\left(\mathbf{y}^{H}\mathbf{A}^{-1}\mathbf{y}\right)^{-1} \sim \mathcal{CW}_{1}\left(m-n+1,\left(\mathbf{y}^{H}\mathbf{\Sigma}^{-1}\mathbf{y}\right)^{-1}\right), \tag{59}$$

which has probability density function,

$$\begin{aligned} f_{W}(w) &= \frac{|w|^{m-n+1-1}\operatorname{etr}\left(-\frac{w}{\theta}\right)}{\Gamma(m-n+1)\,|\theta|^{m-n+1}}\\ &= \frac{w^{m-n}e^{-\frac{w}{\theta}}}{\Gamma(m-n+1)\,\theta^{m-n+1}},\end{aligned} \tag{60}$$

where

$$\theta=\left(\mathbf{y}^{H}\mathbf{\Sigma}^{-1}\mathbf{y}\right)^{-1}. \tag{61}$$

Performing the transformation

$$Z = \frac{2}{\theta} W = 2 \frac{\mathbf{y}^H \mathbf{\Sigma}^{-1} \mathbf{y}}{\mathbf{y}^H \mathbf{A}^{-1} \mathbf{y}}, \tag{62}$$

yields the desired scaled ratio of quadratic forms in equation (58). The probability density function of Z is found to be

$$\begin{aligned} f_Z(z) &= \frac{\theta \left(\frac{z\theta}{2}\right)^{m-n} e^{-\frac{z}{2}}}{\Gamma(m-n+1)\, 2\theta^{m-n+1}} \\ &= \frac{z^{m-n+1-1} e^{-\frac{z}{2}}}{\Gamma(m-n+1)\, 2^{m-n+1}} \\ &= \frac{z^{\frac{r}{2}-1} e^{-\frac{z}{2}}}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}}, \end{aligned} \tag{63}$$

where

$$r = 2(m-n+1), \tag{64}$$

which is a central Chi-squared distribution with $2(m-n+1)$ degrees of freedom.

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